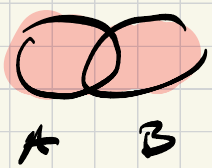
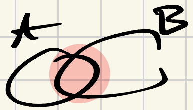
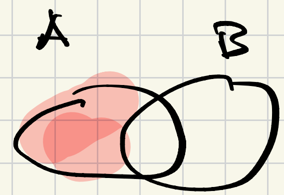
Analysis I Lecture 2

Last time:

Introduced Language of sets:

- $a \in A$ belongs to $\qquad a \notin A$ doesn't belong to
- $B \subseteq A$ subset $\qquad B \not\subseteq A$ not a subset
- $A \cup B$ unions 

$A \cap B$ intersections 

$A \setminus B$ complements 

Sets of numbers:

$\mathbb{N} = \{0, 1, 2, \ldots\}$ - non-negative integers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ all integers

$\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{Z}^* \right\}$$

We have:

Today

Later in the course

$$\begin{array}{ccccccccc} & & a & & \frac{a}{1} = a & & & & \\ & & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} & & & & \\ \mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R} & \subset & \mathbb{C} \\ & & & & & & \text{Real numbers} & & \text{complex numbers} \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \mathbb{N}^* & \subset & \mathbb{Z}^* & \subset & \mathbb{Q}^* & \subset & \mathbb{R}^* & \subset & \mathbb{C}^* \end{array}$$

$\mathbb{N} \not\subset \mathbb{Z}^*$ since $0 \in \mathbb{N}$, $0 \notin \mathbb{Z}^*$

$\mathbb{Z} \not\subset \mathbb{Q}^*$ $\quad 0 \in \mathbb{Z}$, $0 \notin \mathbb{Q}^*$

Today we will discuss **REAL NUMBERS**

Intuitive definition via decimal expansion

$$\underbrace{154387}_{\text{finitely many}}.\underbrace{05321899\ldots}_{\text{possibly infinite}}$$

NEED TO BE MORE CAREFUL

Proposition $0.\overline{9} = 1$

$0.9999\ldots$

Proof:

$$0.\overline{9} \cdot 10 = 9.999\ldots = 9.\overline{9}$$

$$= 9 + 0.\overline{9}$$

$0.\overline{9}$ is a solution of equation:

$$x \cdot 10 = 9 + x$$

Let's solve it :

$$x \cdot 10 = 9 + x$$

$$\Downarrow$$

$$10x - x = 9$$

$$\Downarrow$$

$$9x = 9$$

$$\Downarrow$$

$$x = 1$$

If there is a real number represented by 0,9999...

It actually should be equal to 1

Sum and product notation:

Let $\ell < k$ then we denote

Integers

$$\sum_{i=\ell}^{k} a_i = a_\ell + a_{\ell+1} + \dots + a_{k-1} + a_k$$

$$\prod_{i=\ell}^{k} a_i = a_\ell \cdot a_{\ell+1} \cdot \dots \cdot a_{k-1} \cdot a_k$$

Note: • l, k could be negative.

• l, k could be infinite

Example • $\displaystyle\sum_{i=-1}^{3} 2 = \underset{a_{-1}}{\overset{\shortparallel}{2}} + \underset{a_0}{\overset{\shortparallel}{2}} + \underset{a_1}{\overset{\shortparallel}{2}} + \underset{a_2}{\overset{\shortparallel}{2}} + \underset{a_3}{\overset{\shortparallel}{2}} = 10$

$a_i = 2$

• $\displaystyle\prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n = n!$

$a_i = i$

• $\displaystyle\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + a_3 + \ldots$

Alternative "proof":

Prop. $0.\overline{9} = 1$

$$0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

Preview of later course

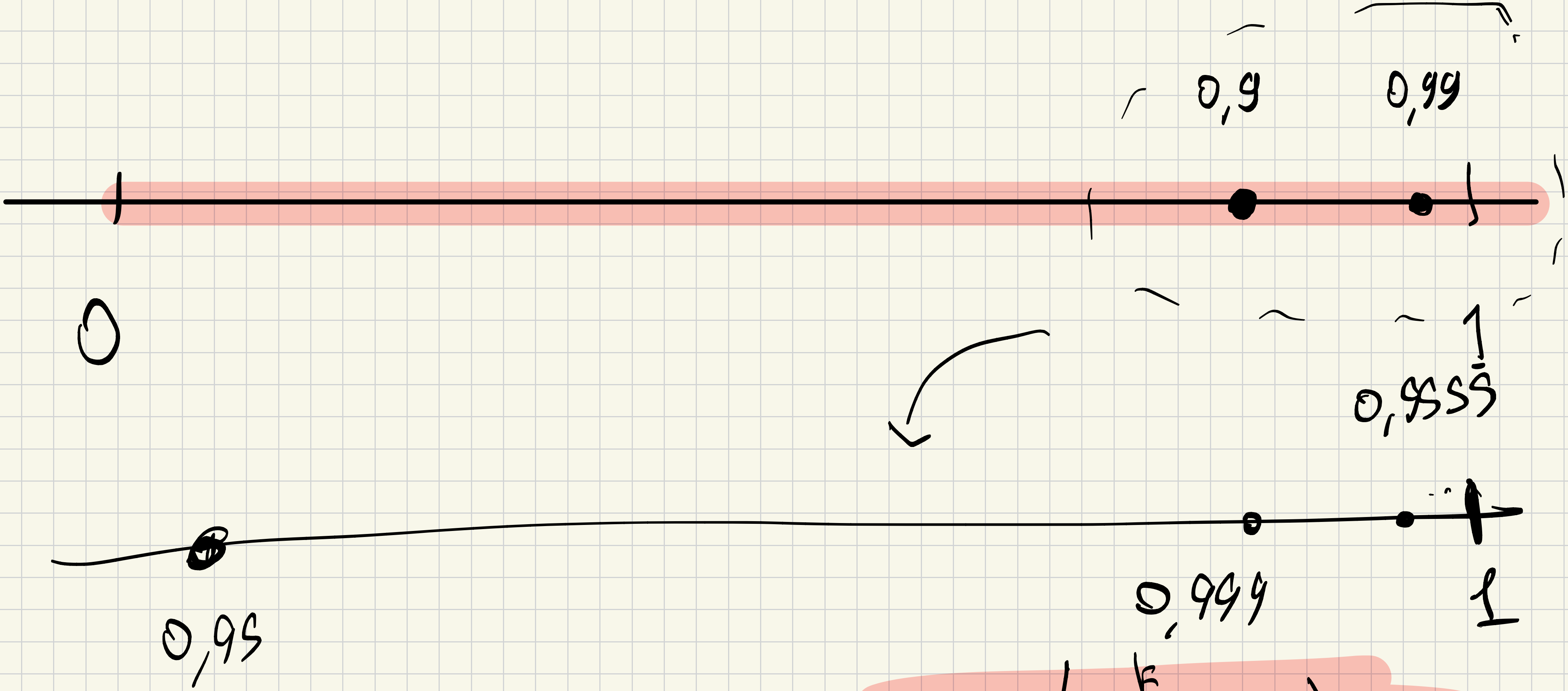
$$= \sum_{i=1}^{\infty} \frac{9}{10^i}$$

$$a_i = \frac{9}{10^i}$$

How do we make sense of this infinite sum?

Idea: consider sums $\sum_{i=1}^{k} \frac{9}{10^i}$ < 1

but we take k larger and larger



In analytic language: $\lim_{k \to \infty} \left(\sum_{i=1}^{k} \frac{9}{10^i} \right) = 1$

So How to define reals?

We use the trick:

Instead of constructing $\mathbb{R}$

we list its properties:

1. $\mathbb{Q} \subset \mathbb{R}$

2. $\mathbb{R}$ is an ORDERED FIELD

3. $\mathbb{R}$ satisfies "The INFIMUM AXIOM"

$\mathbb{R}$ is a FIELD!

We can do arithmetic operations:

- Add, multiply, substract, divide by non-zero number

- With usual properties!

Commutativity

$$x+y = y+x \quad , \quad x \cdot y = y \cdot x$$

distributivity $$x(y+z) = x \cdot y + x \cdot z$$

<u>Example</u>

- In $\mathbb{Z}$ we add, multiply and substract

But we can not divide!

- $\mathbb{N}$ does not have a substraction, $2-5$ is not a natural number

$$x \cdot (y+z) = x \cdot y + x \cdot z$$

$$x \cdot y = y \cdot x$$

- Example from linear algebra:

$$\mathrm{Mat}_{2\times 2} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{R} \right\}$$

$\circlearrowleft$

A, B they have multiplication

BUT $A \cdot B \neq B \cdot A$.

$\mathbb{R}$ is ordered;

For any $x, y \in \mathbb{R}$ we have

either $x < y$ or $y < x$ or $y = x$

And we want that

$x < y$ iff $x + c < y + c$

$x < y$, $c > 0$ then $x \cdot c < y \cdot c$

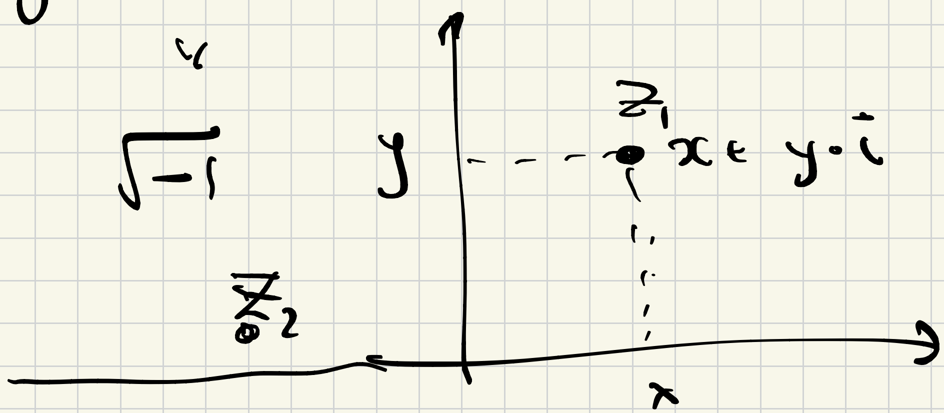
Example: • $\mathbb{Q}$ is an ordered field

• $\mathbb{C}$ field of complex numbers is not ordered.

$x + y \cdot i \qquad x, y \in \mathbb{R}$

$i = \sqrt{-1}$

$\mathbb{R}^2 = \mathbb{C}$



The Infimum axiom

$\{x \mid x \in \mathbb{R}, x > 0\}$

$\parallel$

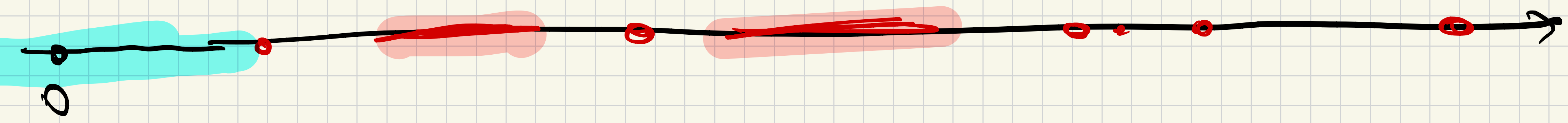
For any $S \subseteq \mathbb{R}_{>0}$

there exists a LARGEST $\ell \in \mathbb{R}$

IN the set

$$\{x \in \mathbb{R} \mid x \leq s \text{ for any } s \in S\}$$

S



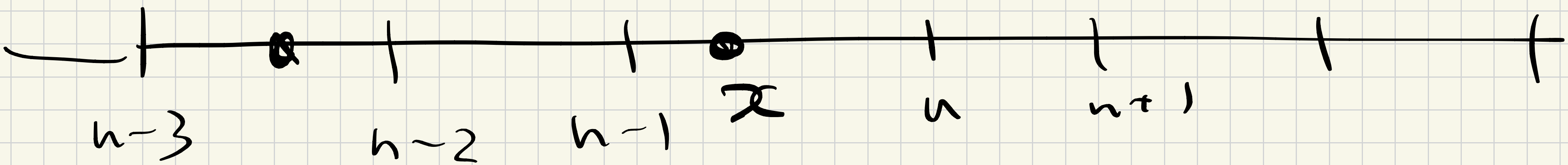
Example

$S = \mathbb{Z}$ then

$$\{x \in \mathbb{R} \mid x \leq s \text{ for any } s \in \mathbb{Z}\}$$

is empty:

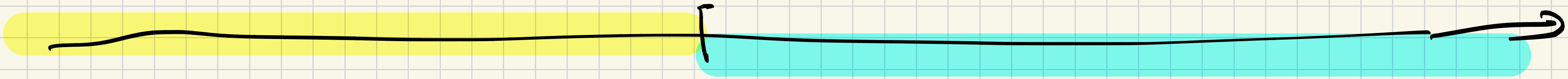
For every $x \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ $n < x$



Example: $\mathbb{Q}$ do not satisfy the infimum axiom:

$$S = \left\{ x \;\middle|\; \begin{array}{l} x \in \mathbb{Q} \\ x^2 > 2 \\ x > 0 \end{array} \right\} \subset \mathbb{Q}_{>0}$$

$x \in \mathbb{Q}$ s.t. x is smaller than any $s \in S$.



$\sqrt{2}$

But the largest such number is $\sqrt{2} \notin \mathbb{Q}$

$$S = \left\{ x \;\middle|\; \begin{array}{l} x \in \mathbb{Q} \\ x^2 > 2 \\ x > 0 \end{array} \right\} \subset \mathbb{Q}_{>0}$$

for any rational number y s.t.
$y < s$ for any $s \in S$
there exists a rational number
y' s.t. $y' < s$ for any $s \in S$ and
$y' > y$.

Instead of constructing $\mathbb{R}$ we list its properties:

1. $\mathbb{Q} \subset \mathbb{R}$
2. $\mathbb{R}$ is an ORDERED FIELD
3. $\mathbb{R}$ satisfies "The INFIMUM AXIOM"

Theorem

The set of REAL numbers exists and is unique.